An Approach Robust Nonlinear Model Predictive Control with State-dependent Disturbances via Linear Matrix Inequalities

Nguyen Thanh Binh, Nguyen Anh Tung, Dao Phuong Nam, Cao Thanh Trung

Abstract—**The issue of nonlinear model predictive control has always been a topic of much concern. We will propose a new approach to robust nonlinear model predictive control to class of nonlinear model system with input constraint under state-dependent disturbances. The considered class of model is separated into linear part at current state, nonlinear part and state-dependent disturbances which are assumed to have their bound. The state-feedback control law is obtained by that solving optimization problem of upper bound of infinite horizon cost function with input constraint via LMIs. In this paper, in order to guarantee robust stability, the proposed approach must generates feasible regions which ensures the existence of a solution and stable region bounded by that. Moreover, these regions are able to contract after every sampling time to proof the robust stability of the system. The simulation results demonstrate the good performance of the proposed approach to RNMPC.**

Keywords – **Linear Matrix Inequalities, Robust Nonlinear Model Predictive Control, Feasibility Region.**

I. INTRODUCTION

Model predictive control is a method possessing some advantages in design of control normal and low dynamic systems. The cost function showing performance of system will be optimized to compute a sequence of optimal control inputs from current to future state, but only the first value input is applied to control the system and the rest of that is eliminated, thus this work will be iterated at each sampling time. Moreover, MPC can analyze some constraint of the system such as state, input and output which cause some drawbacks for several other control approaches. It is clear that the input constraint is important in control design because it relates to the real system.

In the 1970s, MPC was presented the first time to control the linear system and still is researched today. Some authors such as Rawlings [1], Allowger [2,3], Mayne [1,3] and Slotine have researched and progressed MPC for the nonlinear model system with new theories like nominal model, tube and quasi-infinite horizon from the 1990s. In the 21[th] century, researchers have improved MPC to become robust MPC for nonlinear system with additive uncertainties and state-dependent disturbances. Rawlings, Mayne [1] have applied the tube, nominal model and contraction theory, Min-Max theory was proposed by Raimondo to control the nonlinear model. The theories and techniques LMIs presented by Boyd [4] was employed by Kothare [5] to become MPC for the linear system. In recent years, the techniques LMIs have conducted more researches to analyze MPC for the nonlinear system like what Wu, D.jia and Bigdeli [7] have done.

In this paper, we propose a new approach to robust model predictive control to class of nonlinear system under state-dependent disturbances. To our limited knowledge, there is few papers researching into robust nonlinear model predictive control. The considered class of model is separated into linear part at current state, model mismatch and state-dependent disturbance which are assumed to have their bound. Firstly, the proposed optimization problem is quadratic function, upper bound of infinite horizon cost function and solved by technique LMIs – Boyd [4] with input constraint. In order to guarantee robust stability, the optimization problem generates feasible regions which ensures the existence of a solution and stable region bounded by that. The state-feedback control law is obtained by that solving optimization problem of upper bound of infinite horizon cost function via LMIs. This control law is able to contract upper bound to zero, thus it is clear that cost function is also direct to zero and state, input convert to origin, in that the proposed approach ensures robust stability. We will apply the proposed theory to the three-dimension Inverted Pendulum which includes a trolley moving in horizon plane and connecting load via insignificant mass hard bar and assumption of that wind impact on the load from three directions. The objective is that the load is balanced and the trolley moves from any position to origin. Considered system is underactuated, in that two inputs will control four states with their interaction. Therefore, it can be extended to many nonlinear models under state-dependent disturbances.

II. PROPOSED APPROACH

Continuous System under state dependent disturbance:

$$\frac{dx}{dt} = f_c\,(x,u) + h_c\,(x)d \tag{1}$$

Where $x(t) \in R^n$ is state vector, $u(t) \in R^m$ is input vector. $f_c(\bullet,\bullet)$ is nonlinear and continuous differentiable function, $f_c(0,0)$. $g(\bullet)$ is nonlinear and continuous function, $g(0)=0$. $d(t) \in R^q$ is unmeasured external disturbance.

Assumption 1: The external disturbances are in $L_\infty - space$

$$\|d\|_\infty \le d_{\max} \tag{2}$$

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Using forward difference Euler approximation to discretize the system (1), we have a predictive model:

$$x(k+1) = x(k) + T_s f_c(x,u) + T_s h_c(x) d \quad (3)$$

$$x(k+1) = f(x,u) + \Delta \quad (4)$$

The considered state space model will be separated into linear part which is linearized at each measured time, mismatch model and state-dependent disturbances

$$x(k+1|k) = Ax(k|k) + Bu(k) + \tilde{f} + \Delta \quad (5)$$

Where linearization: $A = \left.\frac{\partial f}{\partial x}\right|_{x(k),u(k-1)}$, $B = \left.\frac{\partial f}{\partial u}\right|_{x(k),u(k-1)}$,

mismatch term: $\tilde{f} = f(x(k),u(k)) - Ax(k) - Bu(k)$,

disturbance term: $\Delta = h(x(k|k))d(k|k)$

We can assume that:

$$\|\Delta\| \le \gamma \|x\| \quad given\, \gamma > 0 \quad (6)$$

$$\|\tilde{f}\| \le \delta \|x\| \quad given\, \delta > 0 \quad (7)$$

Remark 1: $\Delta \,\&\, \tilde{f}$ in condition (6,7) are the results of disturbances, mismatch multiplied with sampling time, so sampling time respectively, so sampling time may be selected to satisfy condition (6,7). Both conditions are necessary to generate feasible regions to guarantee a solution in *Theorem 1*. In this paper, the symbol $\|\bullet\|$ is the Euclid norm.

Infinite horizon quadratic cost function:

$$J(k) = \sum_{i=0}^{\infty} [x(k+i|k)^T Qx(k+i|k) + u(k+i|k)^T Ru(k+i|k)] \quad (8)$$

The considered input constraint

$$\|u\| < u_{\max} \quad (9)$$

Where $x(k+i|k)$, $u(k+i|k)$: predicted state and control action in step k, $Q = Q^T > 0$, $R = R^T > 0$

Proposed state feedback control law at step k:

$$u(k+i|k) = F_k x(k+i|k) \quad (10)$$

Proposed Lyapunov function:

$$V(k|k) = x(k|k)^T P_k x(k|k) \quad (11)$$

Lyapunov function is chosen to satisfy the robust condition in [5]:

$$\begin{aligned} &V(k+i+1|k) - V(k+i|k) \le \\ &-x(k+i|k)^T Qx(k+i|k) - u(k+i|k)^T Ru(k+i|k) \end{aligned} \quad (12)$$

Summing (12) from $i = 0 \to i = \infty$

$$J(k) \le x(k|k)^T P_k x(k|k) = V(k|k) \quad (13)$$

Upper bound of cost function (8) is $V(k|k)$ (11). We minimize upper bound of cost function (8) under robust condition (12) by using a problem of LMIs.

Theorem 1: (Robust Condition)

Considering $x(k) = x(k|k)$ which is measured state at sampling time k. $P_k, F_k, H_k, \varepsilon$ are together solution of matrix inequalities (14-18):

$$P_k = P_k^T \ge 0, H_k = H_k^T \ge 0, \varepsilon \ge 0 \quad (14)$$

$$\varepsilon\left[2(\gamma+\delta) + (\gamma^2+\delta^2)\right] I \le H_k \quad (15)$$

$$P_k - \varepsilon I \le 0 \quad (16)$$

$$(A+BF_k)^T P_k (A+BF_k) + Q + F_k^T RF_k + H_k - P_k \le 0 \quad (17)$$

$$\|F_k x\| \le u_{\max} \quad (18)$$

Then, robust condition (12) is true under control law (10) with Lyapunov function (11).

Proof:

Substitute (6) into (12):

$$\begin{aligned} &\left[Ax(k+i|k) + Bu(k+i|k) + \tilde{f} + \Delta\right]^T \times P_k \\ &\times\left[Ax(k+i|k) + Bu(k+i|k) + \tilde{f} + \Delta\right] \\ &-(k+i|k)^T P_k x(k+i|k) \\ &\le -x(k+i|k)^T Qx(k+i|k) - u(k+i|k)^T Ru(k+i|k) \end{aligned} \quad (19)$$

Substitute control law (10) into (19):

$$\begin{aligned} &\left[(A+BF_k)x(k+i|k) + \tilde{f} + \Delta\right]^T \times P_k \\ &\times\left[(A+BF_k)x(k+i|k) + \tilde{f} + \Delta\right] \\ &-x(k+i|k)^T P_k x(k+i|k) \\ &+x(k+i|k)^T (Q + F_k^T RF_k) x(k+i|k) \le 0 \end{aligned} \quad (20)$$

Chose matrix $H_k = H_k^T \ge 0$:

$$\begin{aligned} &2\left((A+BF_k)x(k+i|k)\right)^T P_k (\tilde{f}+\Delta) \\ &+(\tilde{f}+\Delta)^T P_k (\tilde{f}+\Delta) \le x^T H_k x \end{aligned} \quad (21)$$

Then inequality (20) is equivalent to:

$$\begin{aligned} &x(k+i|k)^T [(A+BF_k)^T P_k (A+BF_k) \\ &+Q + F_k^T RF_k + H_k - P_k] x(k+i|k) \le 0 \end{aligned} \quad (22)$$

We evaluate inequality (20):

$$\begin{aligned} &2\left((A+BF_k)x\right)^T P_k (\tilde{f}+\Delta) + (\tilde{f}+\Delta)^T P_k (\tilde{f}+\Delta) \\ &\le 2x^T \left\|(A+BF_k)^T P_k\right\| (\gamma+\delta) x + x^T \lambda_{\max}(P_k)(\gamma^2+\delta^2) x \\ &= x^T \lambda_{\max}(P_k)\left[2\frac{\left\|(A+BF_k)^T P_k\right\|}{\lambda_{\max}(P_k)}(\gamma+\delta) + (\gamma^2+\delta^2)\right] x \end{aligned} \quad (23)$$

$$\begin{aligned} &\left\|(A+BF_k)^T P_k\right\|^2 \le \left\|(A+BF_k)^T P_k^{0.5}\right\|^2 \left\|P_k^{0.5}\right\|^2 \\ &= \left\|(A+BF_k)^T P_k (A+BF_k)\right\| \lambda_{\max}(P_k) \end{aligned} \quad (24)$$

From (22)

$$\begin{aligned}&(A+BF_k)^T P_k (A+BF_k) \le P_k \\ \Rightarrow &\left\|(A+BF_k)^T P_k (A+BF_k)\right\| \le \lambda_{\max}(P_k)\end{aligned} \tag{25}$$

Substituting (25) into (24):

$$\frac{\left\|(A+BF_k)^T P_k\right\|}{\lambda_{\max}(P_k)} \le 1 \tag{26}$$

From (22), (23) and (24) we have robust condition under matrix inequalities (15),(16) and (17). Inequality (18) is input constraint (9). ■

Lemma 1: (Invariant set)

Let P_k, F_k from *Theorem 1*. The system (5) under control $u(k+i\,|\,k) = F_k x(k+i\,|\,k) \forall i \ge 0$ at each step k. Let $C = \left\{x \in R^n \,|\, x^T P_k x \le x(k\,|\,k)^T P_k x(k\,|\,k) = \alpha\right\}$, $x(k+i\,|\,k) \in C \forall i \ge 0$. If state value is in C, the all next step state values are still in C.

Proof:

Because P_k, F_k satisfy *Theorem 1*, the robust condition (12) is considered:

$$\begin{aligned}&x(k+i+1\,|\,k)^T P_k x(k+i+1\,|\,k) \\ &< x(k+i\,|\,k)^T P_k x(k+i\,|\,k) \\ &\forall i \ge 0, x(k+i\,|\,k) \ne 0\end{aligned} \tag{27}$$

$$\begin{aligned}&\Rightarrow x(k+i\,|\,k)^T P_k x(k+i\,|\,k) < x(k\,|\,k)^T P_k x(k\,|\,k) = \alpha \\ &\forall i \ge 1, x(k\,|\,k) \ne 0\end{aligned} \tag{28}$$

$$\begin{aligned}&\Rightarrow x(k+i\,|\,k)^T P_k x(k+i\,|\,k) < x(k\,|\,k)^T P_k x(k\,|\,k) = \alpha \\ &\forall i \ge 1, x(k\,|\,k) \ne 0\end{aligned} \tag{29}$$

$$\Rightarrow x(k+i\,|\,k) \in C \quad \forall i \ge 0 \tag{30}$$

■

Theorem 2: (Minimization of upper bound)

The optimization problem:

$$\underset{T_k, L_k, \eta, E_k}{Min}\; x(k)^T T_k^{-1} x(k) \tag{31}$$

Subject to

$$T_k = T_k^T \ge 0; E_k = E_k^T \ge 0; \eta \ge 0 \tag{32}$$

$$E_k\left[2(\gamma+\delta)+(\gamma^2+\delta^2)\right] \le \eta I \tag{33}$$

$$T_k \ge \eta I \tag{34}$$

$$\begin{bmatrix} T_k & (AT_k+BL_k)^T & T_k^T & L_k^T & T_k^T \\ (AT_k+BL_k) & T_k & 0 & 0 & 0 \\ T_k & 0 & Q^{-1} & 0 & 0 \\ L_k & 0 & 0 & R^{-1} & 0 \\ T_k & 0 & 0 & 0 & E_k \end{bmatrix} \ge 0 \tag{35}$$

$$\begin{bmatrix} \dfrac{u_{\max}^2}{\|x\|_2^2}\eta I & L_k^T \\ L_k & \eta \end{bmatrix} \ge 0 \tag{36}$$

If $T_k, E_k, L_k, \varepsilon$ are the together solution of (31), then $F_k = L_k T_k^{-1}, H_k = E_k^{-1}, P_k = T_k^{-1}, \varepsilon = \eta^{-1}$ will satisfy *Theorem 1* and upper bound of (11) is $x(k)^T T_k^{-1} x(k)$.

Proof:

Set $T_k = P_k^{-1}, L_k = F_k P_k^{-1}, E_k = H_k^{-1}, \eta = \varepsilon^{-1}$

$$(34) \Leftrightarrow P_k^{-1} \ge \varepsilon^{-1} I \Leftrightarrow P_k - \varepsilon I \le 0 \Leftrightarrow (16) \tag{37}$$

$$(33) \Leftrightarrow H_k^{-1}\left[2(\gamma+\delta)+(\gamma^2+\delta^2)\right] \le \varepsilon^{-1} I \Leftrightarrow (15) \tag{38}$$

$$\begin{aligned}&(35) \Leftrightarrow -(AT_k+BL_k)^T T_k^{-1}(AT_k+BL_k) - T_k^T Q T_k \\ &-L_k^T R L_k - T_k^T E_k^{-1} T_k + T_k \ge 0\end{aligned} \tag{39}$$

$$\begin{aligned}&\Leftrightarrow \left(AP_k^{-1}+BF_kP_k^{-1}\right)^T P_k \left(AP_k^{-1}+BF_kP_k^{-1}\right) + P_k^{-1}QP_k^{-1} \\ &+\left(F_kP_k^{-1}\right)^T R\left(F_kP_k^{-1}\right) + P_k^{-1}E_k^{-1}P_k^{-1} - P_k^{-1}P_kP_k^{-1} \le 0 \Leftrightarrow (17)\end{aligned} \tag{40}$$

$$\begin{aligned}&(36) \Leftrightarrow \frac{u_{\max}^2}{\|x\|_2^2}\eta I - \eta^{-1} L_k^T L_k \ge 0 \Rightarrow \|x\|_2^2 \eta^{-2} L_k^T L_k \le u_{\max}^2 I \\ &\Rightarrow \|x\|_2^2 F_k^T F_k \le u_{\max}^2 I \Rightarrow (18)\end{aligned} \tag{41}$$

■

Remark 2: All conditions in *Theorem 2* are linear matrix inequalities or Schur's complement, so it can be solved by Boyd [4]. Parameter γ, δ may be changed after each sampling time to increase convert pace of problem. The YALMIP tool will be apply to find a solution of *Theorem 2*.

Lemma 2: (Feasibility)

The optimal problem in *Theorem 1* is solved at each time k to receive feasible region which will contains all subsequent optimal solution at time $t > k$. Therefore, if constrained region exist and is sought at time k, the next region will be achieved at time $t > k$, the existence of next optimal solution.

Proof:

It can be assumed that we receive feedback matrix control from optimal problem at initial state. The state at time k is bounded by feasible region

$$V(k\,|\,k) = x(k\,|\,k)^T P_k x(k\,|\,k) \le x(k\,|\,k)^T \varepsilon x(k\,|\,k) \tag{42}$$

It can be seen that the solution at time k still satisfy robust condition and feasible region of time $k+1$, thus it points out Lyapunov function of measured state following as:

$$\begin{aligned}&V(k+1\,|\,k+1) = x(k+1\,|\,k+1)^T P_{k+1} x(k+1\,|\,k+1) \\ &\le V(k\,|\,k) = x(k\,|\,k)^T P_k x(k\,|\,k)\end{aligned} \tag{43}$$

It is clear that the measured state at time $k+1$ will also be in feasible region.

$$V\left(k+1 \mid k+1\right) \leq x\left(k \mid \mathrm{k}\right)^{T} \varepsilon x\left(k \mid \mathrm{k}\right) \tag{44}$$

If the feedback matrix control is feasible at time k and initial state, it also exist at time $k+1$. Thus all next state $k+2$, $k+3$ ... will also find a solution of optimal problem. ■

Theorem 3: (Robust stability)

The system (5) under linear feedback control law (10) obtain from *Theorem 2* is robustly asymptotically stable at origin.

Proof:

We note that $(P_k, F_k), (P_{k+1}, F_{k+1})$ is solution obtained in *Theorem 2* at step $k, k+1$ and assume that the optimization problem in *Theorem 2* is feasible at $k=0$. Follow *Lemma 2*, (P_k, F_k) is existed for all $k \geq 1$ and P_k is feasible at $k+1$. We have:

$$\begin{aligned} & x(k+1 \mid k+1)^T P_{k+1} x(k+1 \mid k+1) \\ & < x(k+1 \mid k+1)^T P_k\, x(k+1 \mid k+1) \;\; \forall k \geq 0 \end{aligned} \tag{45}$$

From *Lemma 1*:

$$x(k+1 \mid k)^T P_k\, x(k+1 \mid k) < x(k \mid k)^T P_k\, x(k \mid k) \;\; \forall k \geq 0 \tag{46}$$

Because

$x(k+1 \mid k+1) = f\left(x(k \mid k), u(k \mid k)\right) + g\left(x(k \mid k)\right) d(k \mid k)$

and *Lemma 1* is true $\forall k \geq 0, \forall d(k \mid k)$, from above inequalities we have:

$$\begin{aligned} x(k \mid k)^T P_k\, x(k \mid k) > x(k+1 \mid k+1)^T P_k\, x(k+1 \mid k+1) \\ \forall k \geq 0, x(k \mid k) \neq 0 \end{aligned} \tag{47}$$

Thus Lyapunov function $V(k \mid k) = x(k \mid k)^T P_k\, x(k \mid k)$ is strictly decrease. In addition, from (13), $V(k \mid k) > x(k \mid k)^T Q x(k \mid k)\ \forall x(k \mid k) \neq 0$, we conclude that $x(k) \to 0$ as $k \to \infty$. ■

III. SIMULATION

In this chapter, the proposed method is verified by controlling three dimensional inverted pendulum which includes a trolley moving in horizontal plane and connecting load via a lightweight hard bar. Regarding the external disturbances, it is assumed that wind impacts load from three dimensions Ox, Oy and Oz and wind satisfies the assumption in the previous chapter. Two control inputs u_x and u_y impact the trolley following Ox, Oy to bring system to its stable state at origin.

The Parameters of the three - dimensional inverted pendulum is investigated are mass of trolley $m_t = 5(kg)$, mass of load $m_l = 1(kg)$, length of connecting bar $l = 1(m)$, gravitational accelerator $g = 9.8(m/s^2)$ and sampling time $T_s = 0.1(s)$. The initial states of system are $x_{trolley} = -0.2(m)$, $y_{trolley} = -0.2(m)$, $\varphi_x = 0.2(rad)$, $\varphi_y = 0.3(rad)$, $\dot{x}_{trolley} = 0(m/s)$, $\dot{y}_{trolley} = 0(m/s)$, $\dot{\varphi}_x = 0(rad/s)$ and $\dot{\varphi}_y = 0(rad/s)$. The winds impacting on load are described as random function with amplitude being $0.1(N)$.

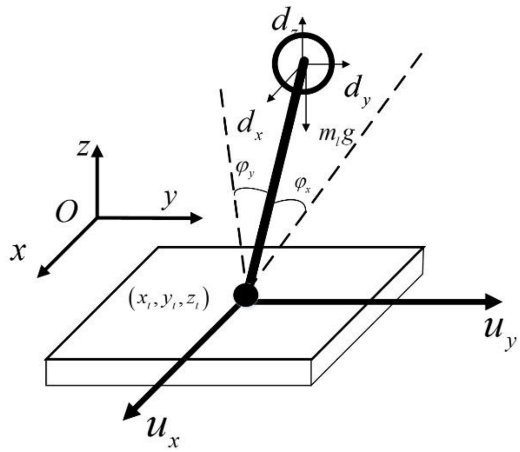

Fig 1. Model of 3D – Inverted Pendulum

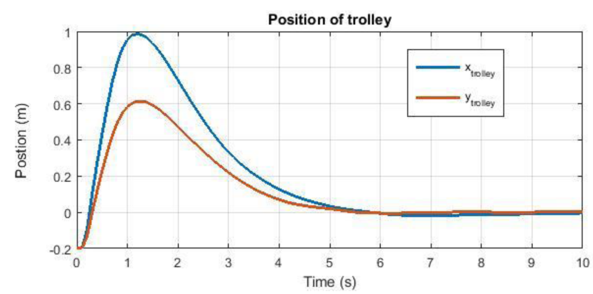

Fig 2. Position of trolley

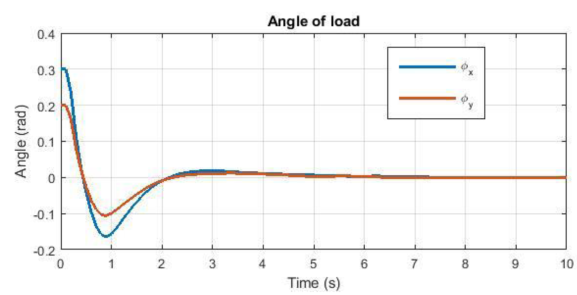

Fig 3. Angle of load

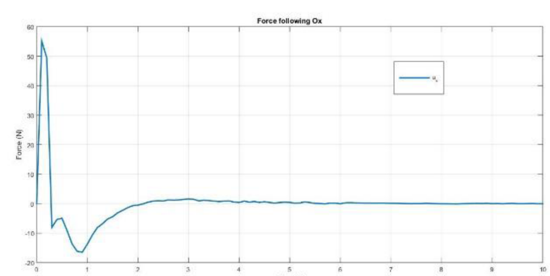

Fig 4. Control input impacts on trolley follow Ox

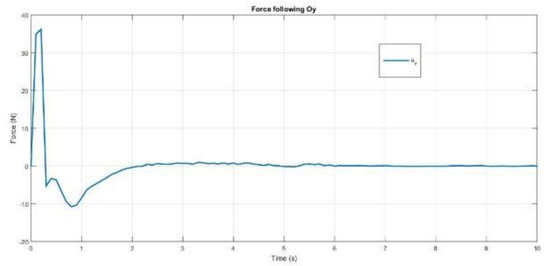

Fig 5. Control input impacts on trolley follow Oy

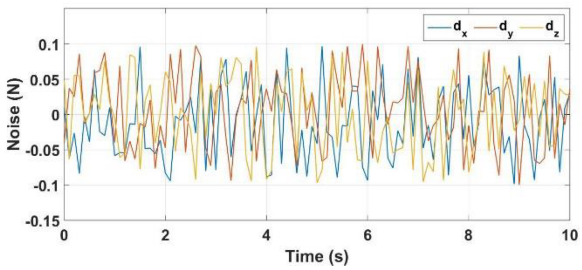

Fig 6. External Disturbances impact on load follow three dimensions

The simulation of 3D-IP investigated by a proposed approach show controller need about 6 seconds to reach its stable state under continuous state-dependent disturbances. The simulation results confirm the good performance of the proposed approach of linear matrix inequalities to robust nonlinear model predictive control.

IV. CONCLUSION

In this paper, the authors propose a new approach to model predictive control to analyze the nonlinear system under state - dependent disturbances which have been a much concerned topic. The considered nonlinear model state space is separated into linear terms to generate linear matrix inequalities, can be solved via LMIs theory - Boyd [4]. The complicated problem of nonlinear optimization is simplified to become linear optimization and linear matrix constraints. Only when this optimal problem can be solved, its solution has good performance which leads nonlinear system to its asymptotic stability.

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